Test for Convergence of Series 2

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ex 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ signs alternate!

prop: series $\sum_{n=1}^{\infty} (-1)^n a_n$ for $a_n \geq 0$ converges if series $\sum_{n=1}^{\infty} |a_n|$ converges (ignore signs) ($\not\leftarrow$: can't go other direction)

notation: $\sum a_n$ = absolutely convergent if $\sum |a_n|$ converges

ex 1 cont.) $a_n = \frac{1}{n}$ $\sum |a_n| = \sum \frac{1}{n} \rightarrow$ diverges (integral test / $p=1$)

nothing can be deduced!

ex 2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\boxed{\text{converges}}$ bc $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (integral test / $p=2$)

$a_n = \frac{1}{n^2}$

thm (alternating series test): suppose (a_n) satisfy

- $a_n \rightarrow 0$
- $a_n > a_{n+1}$ (eventually, $n >> 1$)
- $a_n \geq 0$ (for all a_n)

$\xrightarrow{\text{then}}$ $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

* if fails any of them → don't know *

ex 1 cont.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ satisfies $a_n = \frac{1}{n} \geq 0$ ✓, $a_n \rightarrow 0$ ✓ + decreases ✓

by alternating series test → $\boxed{\text{converges}}$

remark: $\lim \sum \frac{(-1)^{n+1}}{n} = \ln(2)$ (later lectures)

ratio test: suppose (a_n), $a_n \geq 0$ * good for factorials & exponentials *

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$ (next term)
- < 1 → converges (absolute convergence)
- $= 1$ — then $\sum_{n=1}^{\infty} a_n \rightarrow$? ? ?
- > 1 → diverges

ex 1) does $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ ($= a_n$) converges ? ($\frac{4^n}{n!} \rightarrow 0$, might converge)

ratio test needs to compute * $(n+1)! = (n+1)n!$ *

$$\frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} = \frac{4n!}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} \right| = \lim_{n\to\infty} \frac{4n!}{(n+1)!} = \lim_{n\to\infty} \frac{4}{(n+1)} = 0$$

* not canceling → use different test *

since $0 < 1$ → $\boxed{\text{converges}}$

root test: suppose (a_n), $a_n \geq 0$

* root test better than ratio test, if root test fails → don't do ratio test *

if $\lim_{n\to\infty} \sqrt[n]{|a_n|} =$
- < 1 → converges (absolutely convergence)
- $= 1$ — then $\sum a_n \rightarrow$? ? ?
- > 1 → diverges

* integral test better than ratio & root test *

ex 1) does $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converge ? $\leq$ integral test : $\int \frac{1}{x^x}\,dx \to ?$

ratio test : $\frac{(n+1)^{n+1}}{n^n} \to \infty \to ?$ $\left(\frac{n^{n+1}}{n^n} \leq \frac{(n+1)^{n+1}}{n^n} \right.$ diverges $\to$ diverges / needs to give finite value in test to use test $\left.\right)$

* good for power of n *

for root test, we compute

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1 \to \boxed{\text{converges}}$$

$$n! \leq n^n$$
$$2^n \leq n^n \to \frac{1}{n^n} \leq \frac{1}{2^n}$$